Algorithmic Thinking

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The Growth of Functions: An Illustration of Asymptotic Notation

Recall the definition of big-O that we saw in class.

Definition 1 *Let $T(n)$ and $f(n)$ be two functions. We say that $T(n)$ is $O(f(n))$, denoted by $T(n) = O(f(n))$, if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$, we have $T(n) \leq c \cdot f(n)$.*

How do we prove that $T(n) = O(f(n))$ for two functions $T(n)$ and $f(n)$? How do we prove $T(n)$ is not $O(f(n))$ for two other functions? For the former, we have to prove the **witnesses** c and n_0 and establish that $T(n) \leq c \cdot f(n)$. For the latter, we have to show that no matter what choice of $c > 0$ and $n_0 \geq 0$ we make, we can always find an $n \geq n_0$ such that $T(n) \not\leq c \cdot f(n)$. Let's see two examples.

Theorem 1 *$T(n) = n^2 + 2n + 1$ is $O(n^2)$.*

Proof: We need to choose $c > 0$ and $n_0 \geq 0$ such that $n^2 + 2n + 1 \leq cn^2$ for every $n \geq n_0$. While we can try to guess the witnesses c and n_0, we can go about doing so more systematically by first solving the inequality without specific values of c and n_0. Since $2n \leq n^2$ for $n \geq 2$ and $1 \leq n^2$ for $n \geq 1$, it follows that $n^2 + 2n + 1 \leq n^2 + n^2 + n^2$ whenever $n \geq 2$. That is, for $n \geq 2$, we have $n^2 + 2n + 1 \leq 3n^2$. We can now take $n_0 = 2$ and $c = 3$, and that completes the proof. $\square$

Theorem 2 *$T(n) = n^2$ is not $O(n)$.*

Proof: To prove this theorem, we must show that no pair of constants $n_0 \geq 0$ and $c > 0$ exist such that $n^2 \leq cn$for every $n \geq n_0$. That is, we need to show that for <u>any</u> choice of $n_0 \geq 0$ and $c > 0$, we can find <u>at least one</u> value $n \geq n_0$ such that $n^2 \not\leq cn$. We use a proof technique called **proof by contradiction**. We assume that we can find a pair $c > 0$ and $n_0 \geq 0$ such that $n^2 \leq cn$ for all $n \geq n_0$, and arrive at a contradiction. If there exist a pair $c > 0$ and $n_0 \geq 0$ such that $n^2 \leq cn$ for all $n \geq n_0$, then $n^2 - cn \leq 0$, which is equivalent to $n(n - c) \leq 0$ for all $n \geq n_0$. Let n be an integer that is greater than both n_0 and c. Then, n is positive and $n - c$ is positive. Therefore, $n(n - c)$ is positive, which is a contradiction to the assumption that $n(n - c) \leq 0$ for all $n \geq n_0$. $\square$